Engineering Notes

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Estimates of Critical Reynolds Number for a Heated Flat Plate in Water

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Introduction

THE conditions for establishing the onset of turbulence essentially control the design parameters for laminar-flow vehicles. In design calculations, critical Reynolds number can serve as a conservative estimate of the conditions for incipient turbulence. However, estimating critical Reynolds number is not a straightforward task. Computer estimates of critical Reynolds numbers are difficult and expensive. A simplified, analytical analysis that isolates and assesses the influence of important parameters provides a desirable addition to the designer's repertoire of analysis. This paper presents and discusses an extremely simple analysis that appears to be able to predict the critical Reynolds number with surprising accuracy and ease.

This note presents an integration of two approximate analyses that leads to analytical estimates of critical Reynolds number for a heated flat plate in water. Wall shear and heat transfer are calculated from a modified integral approach, first suggested by Zien[1,2] and used to develop a series representation of boundary-layer velocity profile. The series form of the profiles are then used in the modified Dunn-Lin approximation developed by Aroesty et al.,[3] and they result ultimately in estimates of critical Reynolds number. In spite of the drastic simplifications employed to obtain closed-form expressions, numerical results are surprisingly accurate. However, analytical limitations and methods for refinement are discussed.

Analysis

Our model flow is a two-dimensional laminar boundary layer on a heated flat plate with constant density and a variable viscosity. The viscosity-temperature relationship is

$$\mu/\mu_e = 1/[1+\alpha(T-T_e)] \tag{1}$$

where α depends on free-stream temperature T_e and is equal to 0.0152 when $T_e = 67°$F.

The accuracy required for stability calculations is too demanding for traditional approximate boundary-layer solutions, such as the Karman-Pholhausen method. However, Zien[1,2] has revived a method that appears to meet the challenge. This method has demonstrated remarkable insensitivity to assumed profiles and accuracy under the most taxing flow problems, such as flow around a cylinder and the flat plate blow-off problem.

Received Dec. 6, 1978; revision received April 16, 1979.

Index category: Boundary-Layer Stability and Transition.

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Zien derives two integral equations from the momentum equation and two equations from the energy equation. One equation determines the wall parameters such as shear and heat transfer, and the other determines thickness parameters. This same procedure is applied to the heated flat plate problem and the following equations are derived.

a) Wall shear

$$\frac{d}{dx}\int_0^\delta \rho u^2 dy - u_e \frac{d}{dx}\int_0^\delta \rho u dy = -(\mu u_y)_w \tag{2}$$

b) Momentum thickness

$$\int_0^\delta dy' \frac{\partial}{\partial x}\int_0^{y'} \rho u^2 dy - \delta\frac{d}{dx}\int_0^\delta \rho u^2 dy + u_e\delta\frac{d}{dx}\int_0^\delta \rho u dy - \int_0^\delta u dy' \frac{\partial}{\partial x}\int_0^{y'}\rho u dy = \int_0^\delta \mu(T) u_y dy' \tag{3}$$

c) Heat transfer

$$\frac{d}{dx}\int_0^{\delta_T} \rho u (T_e - T) dy = \frac{\mu_e}{Pr}(T_y)_w \tag{4}$$

d) Energy thickness

$$\int_0^{\delta_T} dy' \frac{\partial}{\partial x}\int_0^{y'} \rho u T dy - \delta_T \frac{d}{dx}\int_0^{\delta_T}\rho u T dy + \delta_T T_e \frac{d}{dx} \times \int_0^{\delta_T}\rho u dy - \int_0^{\delta_T} T dy' \frac{\partial}{\partial x}\int_0^{y'}\rho u dy = \frac{\mu_e}{Pr}(T_e - T_w) \tag{5}$$

Equations (2-5) are the primary equations to solve. The procedure is to substitute assumed profiles for the velocity and temperature into the equations and obtain two ordinary differential equations (O.D.E.). This procedure reduces Eqs. (3) and (5) to an O.D.E. for the thickness parameters δ and δ_T and reduces Eqs. (2) and (4) to an O.D.E. for the wall shear $(\mu u_y)_w$ and heat transfer $(\mu_e/Pr)(T_y)_w$. For a more detailed discussion of the derivation refer to Ref. 4.

Series Representation of Velocity Profiles

The series representation of velocity profiles is obtained by developing series solutions of the momentum and energy equations. Employing the traditional boundary layer variables where

$$\eta = (u_e/2\nu x)^{1/2} y; \quad \theta = \frac{T-T_e}{T_w - T_e} = \frac{T-T_e}{\Delta}; \quad u = f'(\eta) \tag{6}$$

The desired series for the velocity and temperature profiles.

$$f'(\eta) = f''(0)\left\{\eta + \left[\frac{\alpha\Delta\theta'(0)}{1+\alpha\Delta}\right]\frac{\eta^2}{2} + f''(0) \times \left[\frac{2\alpha x Pr}{1+\alpha\Delta}\frac{d\Delta}{dx} - (1+\alpha\Delta)\right]\frac{\eta^4}{4!} + \ldots\right\} \tag{7}$$

$$\theta(\eta)=1+\theta'(0)\eta+f''(0)\left(\frac{2\alpha xPr}{1+\alpha\Delta}\ \frac{d\Delta}{dx}\right)\frac{\eta^3}{3!} \tag{8}$$

The interesting aspect of these formulas is that the parameters are conspicuously displayed. These special groupings are suggestive of correlation parameters.[4] Also note that the velocity and temperature profiles depend only on two unknowns $f''(0)$ and $\theta'(0)$. It is the intent of the approximate technique developed in the previous section to determine these unknowns. Below, we obtain results for a flat plate and demonstrate the procedure for calculating critical Reynolds number.

The Heated Flat Plate

The principal aim of the analysis to be presented is to develop closed-form relations for wall shear and heat transfer. Simplification is accomplished by using one-parameter profiles that are surprisingly simple and yet accurate for flat-plate high Prandtl number flow, namely

$$u=u_e\frac{y}{\delta}=\frac{\tau_w}{\mu_e}y$$

$$T=T_w+(T_e-T_w)\frac{y}{\delta_T}=T_e+\Delta[1-(y/\delta_T)] \tag{9}$$

Substituting the above equations in Eqs. (2-5) and after considerable manipulation, one can arrive at explicit expressions for the wall shear and heat transfer after the wall temperature ($\Delta=T_w-T_e$) has been specified. For example, when Δ is constant, the following results can be determined:

$$f''(0)=0.471(1+\alpha\Delta)\left[1-\left(\frac{16}{9Pr}\right)^{1/3}\left(1-\frac{\ln(1+\alpha\Delta)}{\alpha\Delta}\right)\right] \tag{10}$$

$$\theta'(0)=1.00 \tag{11}$$

Later, it will be desirable to explore the influence of surface temperature distribution on boundary-layer stability. Anticipating this requirement, we shall derive expressions for wall shear and heat transfer. However, we shall limit our discussion to power-law surface termperature distributions, ($\Delta\sim x^n$). Closed form results are obtained for the complete system by exploiting $\alpha\Delta\ll 1$ for $\Delta\approx 30°$F. These results are shown below.

$$f''(0)=.471(1+\alpha\Delta)\times\left\{1-\left[\frac{48}{(4n+3)(9)Pr}\right]^{1/3}\left[\frac{\alpha\Delta(2n+1)}{4(n+1)}+0(\alpha^2\Delta^2)\right]\right\} \tag{12}$$

where

$$\theta'(0)=-\sqrt{2}\left[\frac{(2n+1)^3}{(4n+3)^2}\ \frac{4}{9}Pr\right]^{1/3} \tag{13}$$

Refer to Table 1 for the favorable comparison of results from exact numerical solutions of the boundary-layer equations to those from the approximate formulas given above.

Stability

In this section we shall integrate the analyses of previous sections with the Dunn-Lin approximation to develop a simple set of formulas for estimating critical Reynolds number. The success of the modified Dunn-Linn approximation has been discussed in earlier works.[3] The equations are repeated here only for completeness.

$$V(c)=0.58/(1-2\lambda) \tag{14}$$

$$V=\frac{-\pi f'''(c)f''(0)f'(c)}{[f''(c)]^3} \tag{15}$$

$$\lambda=0.4\left\{\left[1-\frac{f''(c)}{f''(0)}\right]+0.5\left[1-\frac{\mu(c)}{\mu(0)}\right]\right\} \tag{16}$$

$$(Re^*_\delta)_{\rm critical}\cong\frac{28f''(0)}{(c)^4(1+\alpha\Delta)} \tag{17}$$

where c is the conventional notation for the velocity at the critical layer (i.e., $f'(\eta_c)=c$), $Re^*_{\delta\,\rm critical}$ is the critical Reynolds number based on displacement thickness.

The problem here is to determine the value of η where the condition given by Eq. (14) is satisfied. This value of η is denoted as η_c, the location of the critical layer. Substituting

Table 1 Comparison of results from approximate boundary-layer analysis to exact calculations

$Te=67°$F,	$Pr=7.18$	$f''(0)$ Approximate	$f''(0)$ Exact	$-\theta'(0)$ Approximate	$-\theta'(0)$ Exact	$Re^*_{\delta\,\rm critical}$ Approximate	$Re^*_{\delta\,\rm critical}$ Exact
$n=0$	$\Delta T=0$	0.471	0.468	1.00	0.930	408	514
	$=20$	0.587	0.584		0.951	2814	2231
	$=30$	0.647	0.646		1.01	4217	4341
$n=¼$	$\Delta T=0$	...	...	1.12	...	...	...
	$=20$	0.582	0.585		1.17	2866	2790
	$=30$	0.632	0.645		1.19	7025	6556
$n=½$	$\Delta T=0$	...	...	1.40	...	...	...
	$=20$	0.581	0.583		1.31	4756	3300
	$=30$	0.631	0.641		1.33	9708	9500
$n=1$	$\Delta T=0$	...	...	1.68	...	...	...
	$=20$	0.582	0.582		1.56	4882	3280
	$=30$	0.630	0.640		1.58	15,553	13,271

Eqs. (1), (7), and (8) into Eq. (16) to find $\lambda(\eta)$, and using this together with Eqs. (14) and (15), one can derive an equation for η_c. After η_c is determined, the velocity is calculated using Eq. (7), which in turn is used to calculate critical Reynolds number from Eq. (17). This procedure must be done numerically and iteratively. However, the computational effort is minimal.

Results and Conclusions

The results presented are a comparison of $Re^*_{\delta\,\text{critical}}$ obtained from our approximate analysis, and an exact analysis.[5] However, the exact analysis consists of an exact boundary-layer calculation in conjunction with the modified Dunn-Lin approximation and an accurate numerical procedure for satisfying the condition expressed by Eq. (14). Therefore, the comparison measures the accuracy of the simplified boundary-layer analysis, the series representation of velocity profile, and the series solution of Eq. (14).

Table 1 shows the accuracy of the approximate analysis of constant surface temperature. The difference between the two results is less than 17%. These inaccuracies can be reduced by refining the temperature profile in the heat transfer calculation.

Also shown in Table 1 is the comparison of results for variable surface temperature. In general, the approximate method provides accurate results; the maximum error does not exceed 30%. More detailed results and comparisons are presented in Ref. 4.

In conclusion, we would like to emphasize the goals and the accomplishments of the study. The study focused on developing a simple, accurate method for calculating critical Reynolds number. Up to now, no such an analysis existed. The principal results demonstrated that the present analysis can be used to accurately estimate critical Reynolds number on a flat plate for a variety of surface temperature distributions without the aid of a computer. The method requires the numerical solution of one algebraic equation; all other relationships are given in closed form. Since computational time and effort are trivial, and the majority of the relationships are given in closed form, the integration of this scheme into an eleborate low drag system design procedure appears desirable and feasible. After the effects of pressure gradients have been incorporated, the method is well-suited for drag optimization studies.

Acknowledgments

The author would like to acknowledge the suggestions and ideas given by his colleagues, Drs. J. Aroesty and C. Gazley, Jr., and a special thanks to C. D. Harris.

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Calculation of Flow in the Tail Region of a Body of Revolution

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Introduction

THE drag of a body of revolution in an incompressible viscous fluid is generally calculated with either the formula of Young[1] or Granville.[2] These formulas require velocity profile shape and momentum parameters at the tail which, for low fineness ratio bodies, are not adequately evaluated using classical boundary-layer theory (see Patel and Guven[3]). This deficiency of boundary-layer analysis motivated a search for a more accurate method for calculating aft body turbulent flow.

The primary approximation used for the resulting method is the neglect of viscous and Reynolds stresses in the rotational flow layer, except for a thin region next to the surface. For airfoils, this approximation is justified near the trailing edge by the rational asymptotic theory of Melnik and Chow.[4] The outer layer in their triple deck model is treated as inviscid and rotational. For bodies of revolution, another justification for neglecting shear stress exists. The contraction of vortex filaments provides a mechanism for vorticity change which dominates the change due to shear stress. A secondary approximation used is equating the vorticity to the reciprocal slope of the velocity profile, a common practice for shear layers with small streamline curvature. The approximation is not essential, since computational methods exist for solution without its introduction.

In the following, the region where the flow is considered to be rotational but without shear stress will be called the calculation region. Surrounding regions are the upstream boundary layer, outer potential flow, and inner viscous surface layer. The boundary conditions for the calculation region can be obtained by an iterative match to the flow calculations for these other regions. Our investigation, however, did not attempt this more extensive problem of computing the entire flowfield; in order to check the calculation method, experimentally measured boundary conditions were used for the outer and upstream borders. The flow was matched to an inner viscous surface layer calculation in a very rudimentary sense by assuming a shape for the velocity profile in that layer.

The Dominant Vorticity Change Mechanism

Consider the material derivative of the vorticity ω

$$\frac{D\omega}{Dt} = \omega \cdot \nabla v + \nu \nabla^2 \omega \tag{1}$$

The first term is associated with vortex filament stretching in the velocity field v, and shear stress is the mechanism responsible for the second term. For laminar flow, ν is the kinematic viscosity. For turbulent flow, consider ν to be an analogous parameter associated with the Reynolds stress.

For axisymmetric flow with axial and radial coordinates x

Received Feb. 20, 1979; revision received July 12, 1979.

Index categories: Boundary Layers and Convective Heat Transfer—Turbulent; Computational Methods; Hydrodynamics.

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